A. Nature of roots of a quadratic equation.

The nature of the roots of a quadratic equation $ax^2 + bx + c = 0$, $a \neq 0$ depends upon the value of the expression $\Delta = b^2 - 4ac$, called the discriminant. The following cases arise (for $a, b, c$ rational numbers).

- **Case I**: When $\Delta > 0$ and not a perfect square, the roots are unequal and irrational.
- **Case II**: When $\Delta > 0$ and a perfect square, the roots are rational and unequal.
- **Case III**: when $\Delta = 0$, the roots are rational and equal.
- **Case IV**: when $\Delta < 0$, the roots are unequal and imaginary.

**Example 1:**
Write the nature of roots of $2x^2 + 3x - 5 = 0$

*Solution:*
Comparing the given equation with $ax^2 + bx + c = 0$, we get $a = 2$, $b = 3$ and $c = -5$.

Now

\[
\Delta = b^2 - 4ac \\
= (3)^2 - 4 \cdot 2 \cdot (-5) \\
= 9 + 40 \\
= 49 \\
=(7)^2 > 0.
\]
Here we got the final result as a perfect square and positive. So it indicates that the given equation must give two roots which are rational and unequal.

**Example 2:**
Write the nature of roots of $2x^2 + 3x - 4 = 0$

*Solution:*
Comparing the given equation with $ax^2 + bx + c = 0$, we get $a = 2$, $b = 3$ and $c = -4$.

Now

\[
\Delta = b^2 - 4ac \\
= (3)^2 - 4 \cdot 2 \cdot (-4) \\
= 9 + 32 \\
= 41
\]
= (3)^2 - 4 \cdot 2 \cdot (-4) \\
= 9 + 32 \\
= 41 > 0.

Here we got the final result as a non-perfect square and positive. So it indicates that the given equation must give two roots which are irrational and unequal.

Example 3:
Write the nature of roots of \(2x^2 + 3x + 5 = 0\)
Solution:
Comparing the given equation with \(ax^2 + bx + c = 0\), we get
\[ a = 2, \ b = 3 \text{ and } c = 5. \]
Now
\[ \Delta = b^2 - 4ac \\
= (3)^2 - 4 \cdot 2 \cdot (5) \\
= 9 - 40 \\
= -31 < 0. \]
Here we got the final result as a negative. So it indicates that the given equation must give two roots which are imaginary and unequal.

Example 4:
Write the nature of roots of \(2x^2 - 4x + 2 = 0\)
Solution:
Comparing the given equation with \(ax^2 + bx + c = 0\), we get
\[ a = 2, \ b = -4 \text{ and } c = 2. \]
Now
\[ \Delta = b^2 - 4ac \\
= (-4)^2 - 4 \cdot 2 \cdot (2) \\
= 16 - 16 \\
= 0. \]
Here we got the final result zero. So it indicates that the given equation must give two roots which are rational and equal. That is, the roots are repeated.

Example 5:
If the equation \(x^2 + (k+2)x + 2k = 0\) has equal roots, find the value of \(k\).
Solution:
comparing the given equation with \(ax^2 + bx + c = 0\)
\[ a = 1, \ b = k + 2, \ c = 2k \]
since the roots are equal;
\[ b^2 - 4ac = 0 \]
or \((k + 2)^2 - 4.1.2k = 0\)

or \(k^2 - 4k + 4 = 0\)

or \((k - 2)^2 = 0\)

hence \(k = 2\).

**B. Relation between roots and coefficients.**

The general form of a quadratic equation is

\[ ax^2 + bx + c = 0 \quad (i) \]

On solving, \(x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \)

If \(\alpha\) and \(\beta\) be two roots of the equation \((i)\) such that

\[ \alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad \beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \]

then sum of the roots

\[ = \alpha + \beta \]

\[ = \frac{-b + \sqrt{b^2 - 4ac}}{2a} + \frac{-b - \sqrt{b^2 - 4ac}}{2a} \]

\[ = \frac{-b}{a} \]

\[ = \frac{-(\text{coefficient of } x)}{\text{coefficient of } x^2} \]

and the product of the roots

\[ = \alpha \beta \]

\[ = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \times \frac{-b - \sqrt{b^2 - 4ac}}{2a} \]

\[ = \frac{c}{a} \]

\[ = \frac{\text{constant term}}{\text{coefficient of } x^2} \]
Example:
Without finding the actual roots of \(2x^2 + 3x - 5 = 0\), find its sum of roots and product of roots.
Solution:
comparing to \(ax^2 + bx + c = 0\), we get \(a = 2\), \(b = 3\) and \(c = -5\).
So,
sum of roots = \(-\frac{b}{a}\)
= \(-\frac{3}{2}\).
product of roots = \(\frac{c}{a}\)
= \(-\frac{5}{2}\).
[Note: you can verify the result by actual calculation of roots]

D. class works.
1. Write the nature of roots of the following quadratic equations.
   i) \(x^2 - 2x + 5 = 0\) ii) \(x^2 - 2x - 3 = 0\) iii) \(x^2 - 2x - 5 = 0\) iv) \(x^2 - 2x + 1 = 0\)
2. Find the sum and product of roots of \(3x^2 - 2x - 3 = 0\)
3. Calculate the sum and product of roots of \(x^2 - 2x - 5 = 0\)

Lecture – 26 (50 minutes)
   a) Formation of a quadratic equations.
   b) Quadratic inequalities and their solution sets.
   c) Class works

A. Formation of quadratic equations.

Suppose \(ax^2 + bx + c = 0\) (\(a \neq 0\)) is the required quadratic equation and \(\alpha\) and \(\beta\) are the given roots.

Then the required equation may be written as
\[x^2 - (\text{sum of the roots})x + \text{product of the roots} = 0\]

Thus a quadratic equation can be formed when sum and product of the roots are known.
Example 1:
Form a quadratic equation whose roots are 2 and 3.

Solution:

sum of roots = 2 + 3 = 5
product of roots = 2.3 = 6
so
required quadratic equation is
\[ x^2 -(\text{sum of roots})x + \text{product of roots} = 0 \]
or \[ x^2 - 5x + 6 = 0. \]

Example 2:
Form a quadratic equation whose roots are twice the roots of \[ 3x^2 + 5x + 1 = 0. \]

Solution:

comparing the given equation with \[ ax^2 + bx + c = 0, \] we get \[ a = 3, \ b = 5 \] and \[ c = 1. \]

suppose \( \alpha \) and \( \beta \) are the roots of the given equation. Then

\[
\text{sum of roots}(\alpha + \beta) = -\frac{b}{a} = \frac{-5}{3}.
\]

\[
\text{product of roots}(\alpha \beta) = \frac{c}{a} = \frac{-1}{3}.
\]

According to the question, the roots of the new equation (i.e. required equation) are \( 2\alpha \) and \( 2\beta \).

For this

\[
\text{sum of roots} = 2\alpha + 2\beta = 2(\alpha + \beta) = 2 \cdot \frac{-5}{3} = -\frac{10}{3}.
\]

\[
\text{product of roots} = 2\alpha.2\beta = 4.\alpha\beta = 4 \cdot \frac{-1}{3} = -\frac{4}{3}.
\]

Hence the required equation is

\[ x^2 -(\text{sum of roots})x + \text{product of roots} = 0. \]
or \[ x^2 -\frac{10}{3}x + \frac{-4}{3} = 0 \]
or \[ 3x^2 + 10x - 4 = 0. \]
B. Quadratic inequalities and their solution sets.

If we have a quadratic inequality, we can often solve it if the algebraic expression can be rewritten in factorial form. The following example may make the concept clear.

Example:
Solve the inequality \( x^2 - 3x + 2 \leq 0 \).

Solution:

we have \( x^2 - 3x + 2 \leq 0 \).

or \((x-2)(x-1) \leq 0\).

From this we can say that the number line is divided into three parts cutting at \( x = 1 \) and \( x = 2 \). So the possible inequalities are \( x<1 \), \( 1<x<2 \) and \( x>2 \). One or two of them is the required solution of the given inequality. We need to add equal sign to the required solution because there is \( \leq \) sign which contains equality also.

For \( x < 1 \), the value of \( x - 2 \) is always -ve and \( x-1 \) is also -ve. ... ... ... [a]

So, the product of \( x - 2 \) and \( x - 1 \) is +ve

For \( 1<x<2 \), the value of \( x - 2 \) is always -ve and \( x-1 \) is +ve. ... ... ... [b]

So, the product of \( x - 2 \) and \( x - 1 \) is -ve

For \( x>2 \), the value of \( x - 2 \) is always +ve and \( x-1 \) is also +ve. ... ... ... [c]

So, the product of \( x - 2 \) and \( x - 1 \) is +ve

Since we need the product negative or equal to zero \([x-2](x-1) \leq 0]\); the required solution is of type [b] with equality.

Hence the required solution is \( 1 \leq x \leq 2 \).

This can be summarized int he following table:

<table>
<thead>
<tr>
<th></th>
<th>( x&lt;1 )</th>
<th>( 1&lt;x&lt;2 )</th>
<th>( x&gt;2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x-2 )</td>
<td>-ve</td>
<td>-ve</td>
<td>+ve</td>
</tr>
<tr>
<td>( x-1 )</td>
<td>-ve</td>
<td>+ve</td>
<td>+ve</td>
</tr>
<tr>
<td>( (x-1)(x-2) )</td>
<td>+ve</td>
<td>-ve</td>
<td>+ve</td>
</tr>
</tbody>
</table>

For \( (x-1)(x-2) < 0 \), the middle column gives the solution area.

Hence for \( (x-1)(x-2) \leq 0 \), the solution set is \( 1 \leq x \leq 2 \).

C. Class works.

1. Form a quadratic equation whose roots are squares of the roots of \( x^2 - 2x - 3 = 0 \)
2. Form a quadratic equation whose roots are thrice the roots of \( x^2 - 2x - 5 = 0 \)
3. Solve the inequalities given below.

   i) \( x^2 - 3x + 2 \leq 0 \)  
   ii) \( x^2 - 3x - 4 \geq 0 \)  
   iii) \( 2x^2 - 3x - 2 \leq 0 \)
Lecture – 27 (50 minutes)

a) Exercise of the problems related to quadratic equations.

b) Assignment

A. Exercise of the problems related to quadratic equations.

1. Determine the nature of the roots of the equations

   i) \( x^2 - 4x + 3 = 0 \)  
   ii) \( 2x^2 - 6x + 7 = 0 \)
   iii) \( x^2 - 4x + 1 \)

2. For what values of \( a \) will the equation \( x^2 - (3a - 1)x + 2(a^2 - 1) = 0 \) have equal roots?

3. If the roots of the quadratic equation \( 9x^2 - 6x + k = 0 \) are equal find the value of \( k \).

4. Find the value of \( k \) so that the equation \( 2x^2 + kx - 15 = 0 \) has one root 3.

5. Form a quadratic equation whose roots are the reciprocals of the roots of \( 2x^2 + 5x + 4 = 0 \).

B. Assignment.

1. Find the value of \( k \) so that the equation \( 3x^2 + kx - 2 = 0 \) has roots whose sum is 6.

2. If \( 2x^2 + (4 - k)x - 17 = 0 \) has roots equal but opposite in sign, find the value of \( k \).

3. Find the value of \( k \) so that the equation \( 3x^2 + 7x + 6 - k = 0 \) has one root equal to zero.

4. If the roots of \( ax^2 + bx + c = 0 \) be in the ratio 3:4, prove that \( 12b^2 = 49ac \).

   [Hints: take one root \( 3\alpha \) and other \( 4\alpha \). Then using sum of roots = -b/a, find the value of \( \alpha \) and put that value in the relation of product of roots.]

5. If one root of the equation \( x^2 - px + q = 0 \) be twice the other, show that \( 2p^2 = 9q \).

6. For what values of \( m \), the equation \( x^2 - mx + m + 1 = 0 \) may have its roots in the ratio 2:3?

7. What are the roots of \( ax^2 + bx + c = 0 \).

8. If the roots of \( (a^2 + b^2) x^2 - 2(ac + bd) x + (c^2 + d^2) = 0 \) are equal show that \( ad = bc \).

9. Form the quadratic equation in which roots are 3, -2.

10. Find the quadratic equation whose roots are the reciprocals of the roots of \( 3x^2 - 5x - 2 = 0 \).

11. Form the quadratic equation whose roots will be

    a) \( m \) times the roots of \( x^2 - px + q = 0 \).

    b) greater by \( h \) than the roots of \( x^2 - px + q = 0 \).

    c) the reciprocals of the roots of the equation \( ax^2 + bx + c = 0 \).

12. If \( \alpha \) and \( \beta \) are the roots of \( ax^2 + bx + c = 0 \), find the equation whose roots are \( \alpha \beta^{-1} \) and \( \beta \alpha^{-1} \).